Transforms
-in the context of probability

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Terminology

- PDF : Probability Density Function
- PMF : Probability Mass Function
- Expectation :

$$E[X] = \int_{-\infty}^{\infty} x.f(x).dx$$

Drake and Bertsekas differ....

- The two texts diverge in their definitions of ‘transforms’ in the context of probability
  - Add into that the way I was taught as an Engineering Student!

- Drake goes into transforms more deeply and is closer to my understanding of s-transforms (Laplace)

Bertsekas

$$M_X(s) = E[e^{sX}]$$

$$M(s) = \sum_x e^{s.x}.p_X(x)$$

$$M(s) = \int_{-\infty}^{\infty} e^{s.x}.f_X(x).dx$$

Drake

$$f_x(s) = E[e^{-s.x}]$$

Signs!

The Drake definition is consistent with every other text I can find on the Laplace Transform! (and my engineering education)

Drake Ex.4.1 - Discrete

$$p_x(x) = \begin{cases} \frac{1}{2}, if\ x = 2 \\ \frac{1}{6}, if\ x = 3 \\ \frac{1}{3}, if\ x = 5 \end{cases}$$

The PMF

$$M(s) = \frac{1}{2}.e^{2.s} + \frac{1}{6}.e^{3.s} + \frac{1}{3}.e^{5.s}$$

Drake Ex.4.4-Linear function of RV

- This is an important result used later:
  - If X is a RV, and:

$$Y = a.X + b$$

$$M_Y(s) = E\left[e^{s(a.X+b)}\right] = e^{sb}.E\left[e^{saX}\right] = e^{sb} M_X(sa)$$

Linear property of transforms

Drake Ex.4.5 - Continuous

- Normal Distribution $$f_Y(y) = \frac{1}{\sqrt{2.\pi}}.e^{\frac{-y^2}{2}}$$

$$M_Y(s) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2.\pi}}.e^{\frac{-y^2}{2}}.e^{s.y}.dy$$

$$= \frac{1}{\sqrt{2.\pi}} \int_{-\infty}^{\infty} e^{\frac{-y^2}{2}+s.y} dy$$

$$= e^{\frac{s^2}{2}}.\frac{1}{\sqrt{2.\pi}} \int_{-\infty}^{\infty} e^{-(\frac{y^2}{2})+s.y-(\frac{s^2}{2})} dy$$

$$= e^{\frac{s^2}{2}}.\frac{1}{\sqrt{2.\pi}} \int_{-\infty}^{\infty} e^{\frac{-(y-s)^2}{2}} dy$$

Comes from normalization property.

$$\therefore M_Y(s) = e^{\frac{s^2}{2}}$$

Transform of any Normal with non unity SD and Mean/Expectation

- Using previous 2 results:

$$if\ X = \sigma.Y + \mu$$

- Then the transform of this new Normal distribution is:

$$M_X(s) = e^{s.\mu}.M_Y(s.\sigma) = e^{\left(\frac{\sigma^2.s^2}{2}+\mu.s\right)}$$

From Transforms to Moments

- 'Moment generating functions' : Transforms
  - Different 'moments' are easily computed once we have the transform
  - Proof based on differentiating both sides of the original transform equation

$$M(s) = \int_{-\infty}^{\infty} e^{s.x}.f_X(x).dx$$

$$\frac{d}{ds}M(s) = \frac{d}{ds}\int_{-\infty}^{\infty} e^{sx}.f_X(x).\,dx$$

$$= \int_{-\infty}^{\infty} \frac{d}{ds}.e^{sx}.f_X(x).dx$$

$$= \int_{-\infty}^{\infty} x.e^{sx}.f_X(x).dx$$

Now consider case with s=0

$$\frac{d}{ds}M(s)\bigg|_{s=0} = \int_{-\infty}^{\infty} x.f_X(x).dx = E[X]$$

It can be shown that this holds for the nth differential:

$$\frac{d^n}{ds^n}M(s)\bigg|_{s=0} = \int_{-\infty}^{\infty} x^n.f_X(x).dx = E[X^n]$$

This basically tells us that once we have the transform of the RV, then we can quickly compute the nth moment by taking the nth differential of that transform

Ex.4.6

- From Ex.4.1 $p_x(x)=\begin{cases}\frac{1}{2}, if\ x=2\\ \frac{1}{6}, if\ x=3\\ \frac{1}{3}, if\ x=5\end{cases}$ $M(s)=\frac{1}{2}.e^{2.s}+\frac{1}{6}.e^{3.s}+\frac{1}{3}.e^{5.s}$

$$E[X]=\frac{d}{ds}M(s)\bigg|_{s=0}$$

$$=e^{2s}+\frac{1}{2}.e^{3s}+\frac{5}{3}.e^{5s}\bigg|_{s=0}$$

$$=\frac{19}{6}$$

Transforms Part2
2/1/11

Inversion of Transforms

- Transforms are usually 'reversed' by recognizing a 'form' from a table.
- Ex.4.7

$$M(s) = \frac{1}{4}.e^{-s} + \frac{1}{2} + \frac{1}{8}.e^{4.s} + \frac{1}{8}.e^{5.s}$$

- This is of the form:

$$M(s) = \sum_{x} e^{s.x}.p_X(x)$$

- And so we deduce that Px is a discrete random variable with: $P(X=-1)=\frac{1}{4}$ $P(X=0)=\frac{1}{2}$ $P(X=4)=\frac{1}{8}$ $P(X=5)=\frac{1}{8}$

Transform of a mix of two Dist.

- Ex.4.9
  - Combination of two exponential distributions

$$f_X(x) = \frac{2}{3}.6e^{-6x} + \frac{1}{3}.4e^{-4x}$$

$$M(s) = \int_0^{\infty} e^{s.x}.\left(\frac{2}{3}.6e^{-6x} + \frac{1}{3}.4e^{-4x}\right).dx$$

$$M(s) = \frac{2}{3}\int_0^{\infty} e^{s.x}.6e^{-6x}dx + \frac{1}{3}\int_0^{\infty} e^{s.x}.4e^{-4x}dx$$

So the transform of the sum of two pdfs is just the sum of the transforms of the two pdfs

$$if\ \ f_Y(y) = p_1 f_{x1}(y) + ... + p_n f_{xn}(y)$$

$$then\ M_Y(s) = p_1 M_{x1}(s) + ... + p_n M_{xn}(s)$$

Sum of Independent RVs corresponds to multiplication of their transforms

- Note:
  - Multiplication in the transform domain is equivalent to convolution in the original variable or function
    - In this case the pdfs would be convolved with each other
- X and Y are independent RVs, W=X+Y

$$M_W(s) = E\left[e^{sW}\right] = E\left[e^{s(X+Y)}\right] = E\left[e^{sX}.e^{sY}\right]$$

If s is considered fixed:

$$M_W(s) = E\left[e^{sX}.e^{sY}\right] = E\left[e^{sX}\right].E\left[e^{sY}\right] = M_X(s).M_Y(s)$$

Generalizing:

$$\begin{aligned} &if \\ &W = X_1 + ...X_{n,} \\ &then \\ &M_W(s) = M_{X1}(s).M_{X2}(s)..........M_{Xn}(s) = \prod_n M_{Xn}(s) \end{aligned}$$

Sum of independent Normal RVs

- X and Y are independent normal RVs with:

$$\text{means}: \mu_x \text{ and } \mu_y$$

$$\text{variances}: \sigma_x^2 \,\&\, \sigma_y^2$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}}.e^{\frac{-(x-\mu)^2}{2.\sigma^2}}$$

$$M_X(s) = e^{\frac{\sigma_x^2.s^2}{2}+\mu_x^s}, \quad M_Y(s) = e^{\frac{\sigma_y^2.s^2}{2}+\mu_y^s}$$

$$M_W(s) = e^{\frac{\left(\sigma_x^2+\sigma_y^2\right)s^2}{2}+\left(\mu_x+\mu_y\right)s}$$

The transform is the same as the transform of a single Normal RV
-with the means and variances summed

Sums of Independent RVs

- If X & Y are independent RVs, then the distribution of their sum W=X+Y can be obtained by using the product of the transforms of the distributions of X and Y and then doing an inverse transform

- Alternatively it can be computed directly using convolution….

Discrete Case of Convolution

- X&Y are independent integer valued RVs with PMFs PX(x) and PY(y)

$$P_W(w) = P(X+Y)$$

$$= \sum_{(x,y):x+y=w} P\left(X = x \,\&\, Y = y\right)$$

for w being any integer

$$= \sum_x P(X = x \,\&\, Y = (w-x))$$

$$= \sum_x p_x(x) p_y(w-x)$$

Convolution sum

Continuous case of convolution

- X & Y are independent continuous RVs with PDFs f(x) and f(y) respectively

$$
\begin{aligned}
F_W(w) &= P(W \le w) \\
&= P((X+Y) \le w) \\
&= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{w-x} f_X(x) f_Y(y)\, dy.\, dx \\
&= \int_{x=-\infty}^{\infty} f_X(x) \left[\int_{y=-\infty}^{w-x} f_Y(y).\, dy \right] . dx \\
&= \int_{x=-\infty}^{\infty} f_X(x) . F_Y(w-x)\, dx
\end{aligned}
$$

Find the CDF expression for the summed RVs

Continued…

Continuous convolution cont.

- The PDFof W is then obtained by differentiating the expression for the CDF:

$$f_W(w) = \frac{dF_W}{dw}(w)$$

$$= \frac{d}{dw}\int_{-\infty}^{\infty} f_X(x)F_Y(w-x)\,dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\frac{dF_Y(w-x)}{dw}\,dx$$

$$= \int_{-\infty}^{\infty} f_X(x)f_Y(w-x)\,dx$$

Convolution integral

Ex 4.14

- X&Y are independent and uniformly distributed in the interval [0,1] (and so the height of the pdf is 1). The PDF of W=X+Y :

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx$$

$$f_X(x) = \begin{cases} 1 \; for \; 0 \le x \le 1 \\ 0 \; otherwize \end{cases}$$

$$f_Y(y) = \begin{cases} 1 \; for \; 0 \le (w-x) \le 1 \\ 0 \; otherwize \end{cases}$$

So from w=0 to w=2, small dx increments get summed to produce a triangular distribution